浙江大学
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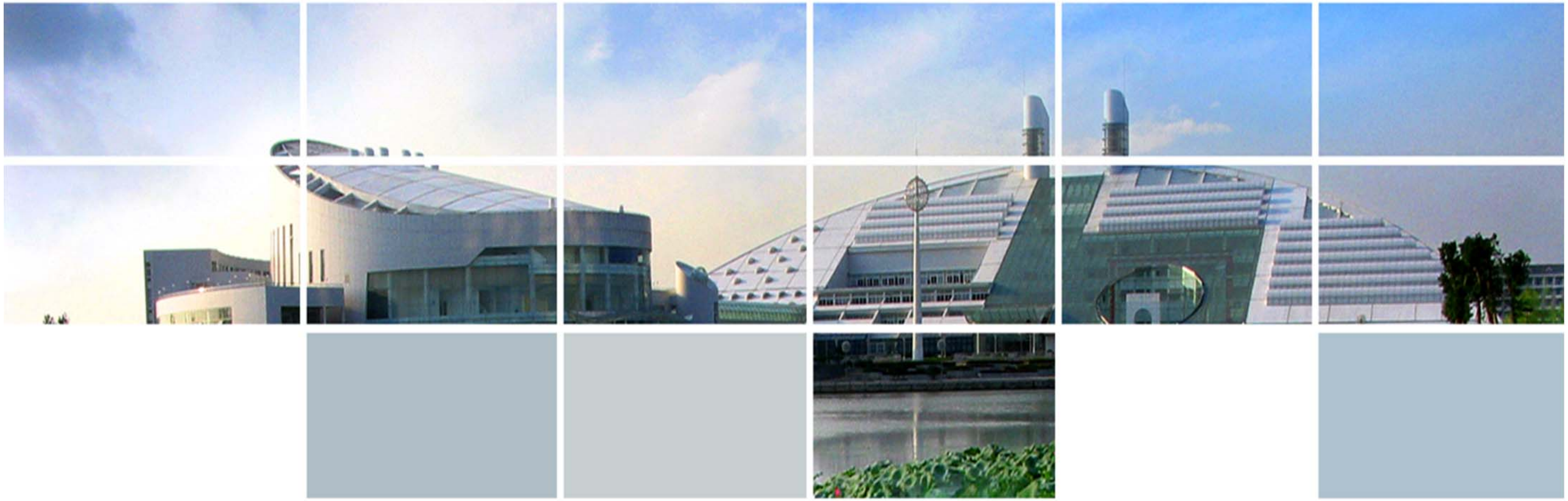

第43讲 两个正态总体的抽样分布

定理三: 设样本$\left(X_1,\cdots,X_{n_1}\right)$和$\left(Y_1,\cdots,Y_{n_2}\right)$分别来自总体$N\left(\mu_1,\sigma_1^{\ 2}\right)$和$N\left(\mu_2,\sigma_2^{\ 2}\right)$,并且它们相互独立.

样本均值分别为$\overline{X},\overline{Y}$;样本方差分别为$S_1^2,S_2^2$.

则可以得到下面三个抽样分布.

(1) $$F=\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2}=\frac{S_1^2}{S_2^2}\bigg/\frac{\sigma_1^2}{\sigma_2^2}\sim F(n_1-1,n_2-1).$$

证明: $$\chi_1^2=\frac{(n_1-1)S_1^2}{\sigma_1^2}\sim\chi^2(n_1-1)$$

$$\chi_2^2=\frac{(n_2-1)S_2^2}{\sigma_2^2}\sim\chi^2(n_2-1)\quad S_1^2,S_2^2 \text{ 独立},$$

$$\therefore F=\frac{S_1^2}{S_2^2}\bigg/\frac{\sigma_1^2}{\sigma_2^2}=\frac{\chi_1^2/(n_1-1)}{\chi_2^2/(n_2-1)}\sim F(n_1-1,n_2-1)$$

(2) $$\frac{\left(\bar{X}-\bar{Y}\right)-\left(\mu_1-\mu_2\right)}{\sqrt{\dfrac{\sigma_1^2}{n_1}+\dfrac{\sigma_2^2}{n_2}}} \sim N(0,1)$$

证明：$\bar{X} \sim N(\mu_1, \dfrac{\sigma_1^2}{n_1}),\ \bar{Y} \sim N(\mu_2, \dfrac{\sigma_2^2}{n_2}),$

$$\Rightarrow \bar{X}-\bar{Y} \sim N(\mu_1-\mu_2, \frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}),$$

$$\Rightarrow \left[\left(\bar{X}-\bar{Y}\right)-\left(\mu_1-\mu_2\right)\right] \Big/ \sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}} \sim N(0,1).$$

（3）当$\sigma_1^2=\sigma_2^2=\sigma^2$时，

$$\frac{\left(\bar{X}-\bar{Y}\right)-\left(\mu_1-\mu_2\right)}{S_w\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}\sim t\left(n_1+n_2-2\right)$$

其中$S_w^2=\dfrac{\left(n_1-1\right)S_1^2+\left(n_2-1\right)S_2^2}{n_1+n_2-2}, S_w=\sqrt{S_w^2}$

证明：当$\sigma_1^2=\sigma_2^2=\sigma^2$时，

由(2) $$\frac{\left(\bar{X}-\bar{Y}\right)-(\mu_1-\mu_2)}{\sigma\sqrt{\dfrac{1}{n_1}+\dfrac{1}{n_2}}}\sim N(0,1)$$

$$\frac{(n_1-1)S_1^2}{\sigma^2}\sim\chi^2(n_1-1),\ \frac{(n_2-1)S_2^2}{\sigma^2}\sim\chi^2(n_2-1)$$

$$\therefore\frac{(n_1-1)S_1^2+(n_2-1)S_2^2}{\sigma^2}\sim\chi^2(n_1+n_2-2)$$

由t分布定义，$\dfrac{(\bar{X}-\bar{Y})-(\mu_1-\mu_2)}{S_w\sqrt{\dfrac{1}{n_1}+\dfrac{1}{n_2}}}$

$$=\frac{(\bar{X}-\bar{Y})-(\mu_1-\mu_2)}{\sigma\sqrt{\dfrac{1}{n_1}+\dfrac{1}{n_2}}}\Bigg/\sqrt{\frac{(n_1-1)S_1^2+(n_2-1)S_2^2}{\sigma^2(n_1+n_2-2)}}$$

$$\sim t(n_1+n_2-2)$$

思考: 若σ^2未知，为什么用S_w^2来估计σ^2， 

而不用S_1^2或S_2^2来估计σ^2呢？

$$E(S_1^2)=\sigma^2 \qquad E(S_2^2)=\sigma^2 \qquad E(S_w^2)=\sigma^2$$

$$D(S_1^2)=\frac{2\sigma^4}{n_1-1} \qquad D(S_2^2)=\frac{2\sigma^4}{n_2-1}$$ —第42讲例2

$$S_w^2=\frac{(n_1-1)S_1^2+(n_2-1)S_2^2}{n_1+n_2-2}$$

从直观来看，S_w^2比S_1^2或S_2^2包含更多σ^2的信息

$$D(S_w^2)=\frac{2\sigma^4}{n_1+n_2-2}$$

比$D(S_1^2)$，$D(S_2^2)$更小.

对于单个正态总体$N(\mu,\sigma^2)$，得到了$\bar{X}$，S^2的分布，用于对μ,σ^2进行推断(区间估计,假设检验).

对于两个独立正态总体$N(\mu_1,\sigma_1^2)$, $N(\mu_2,\sigma_2^2)$，得到了$\bar{X}-\bar{Y},S_1^2/S_2^2$的分布，用于对$\mu_1-\mu_2,\dfrac{\sigma_1^2}{\sigma_2^2}$进行推断.