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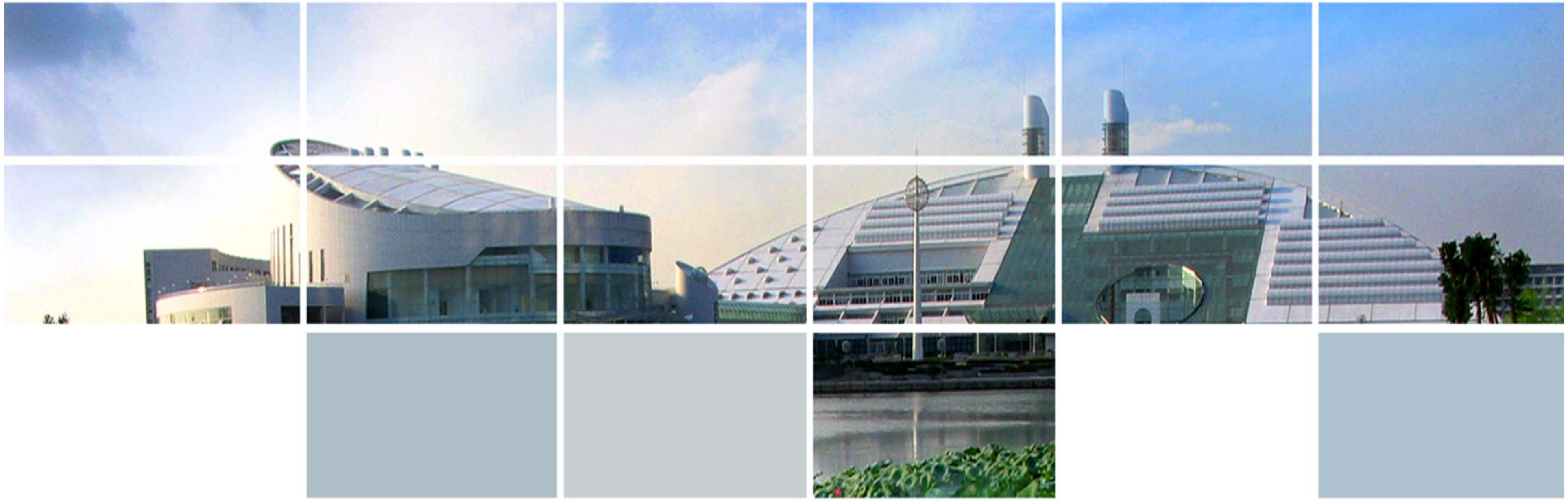

第42讲 单个正态总体的抽样分布

定理一：设总体$X \sim N(\mu, \sigma^2), X_1, X_2, \cdots, X_n$是样本，

样本均值$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$，样本方差$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$.

则 (1) $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$;

(2) $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$且$\bar{X}$与$S^2$相互独立.

(1) 证明: $E(\bar{X}) = E(\frac{1}{n}\sum_{i=1}^{n} X_i) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \mu,$

$$D(\bar{X}) = D(\frac{1}{n}\sum_{i=1}^{n} X_i) = \frac{1}{n^2}\sum_{i=1}^{n} D(X_i) = \frac{\sigma^2}{n},$$

$X_1, X_2, \ldots X_n$独立且都服从正态分布,

而且$\bar{X}$是$X_1, X_2, \ldots X_n$的线性组合

$\Rightarrow \bar{X}$ 服从正态分布，即$\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$.

（2）证略.

例1： 设总体$X \sim N(\mu,\sigma^2), X_1, X_2, \cdots, X_n$是样本，

$$(1)\ \frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{\sigma^2} \sim \chi^2(n-1)$$

$$\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{\sigma^2} = \frac{(n-1)S^2}{\sigma^2}$$

$X_1-\bar{X},\cdots,X_n-\bar{X}$

有一个约束条件

$$\sum_{i=1}^{n}(X_i-\bar{X})=0$$

$$(2)\ \frac{\sum_{i=1}^{n}(X_i-\mu)^2}{\sigma^2} \sim \chi^2(n)$$

$$\frac{\sum_{i=1}^{n}(X_i-\mu)^2}{\sigma^2} = \sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2$$

由定理一(1)知，$\dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1).$

当σ未知时,可用S来代替σ,此时有

定理二: 设总体$X \sim N(\mu,\sigma^2), X_1, X_2, \cdots, X_n$是样本,

样本均值$\bar{X}=\dfrac{1}{n}\sum_{i=1}^{n}X_i$,样本方差$S^2=\dfrac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2.$

则 $\dfrac{\bar{X}-\mu}{S/\sqrt{n}} \sim t(n-1).$

证明: $\dfrac{\overline{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1), \dfrac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$

$\overline{X}$与S^2相互独立,

$$\therefore \frac{\overline{X}-\mu}{S/\sqrt{n}} = \frac{\dfrac{\overline{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\dfrac{(n-1)S^2}{\sigma^2(n-1)}}} \sim t(n-1).$$

例2：设总体X的均值μ,方差σ^2存在.

$\left(X_1,\cdots,X_n\right)$是取自总体$X$的样本,

$\bar{X},S^2$为样本均值和样本方差.

(1) 求$E(S^2)$;

(2)若$X\sim N\left(\mu,\sigma^2\right)$,求$D(S^2)$.

解(1): $$\sum_{i=1}^{n}(X_i-\bar{X})^2=\sum_{i=1}^{n}(X_i^2-2\bar{X}X_i+\bar{X}^2)$$

$$=\sum_{i=1}^{n}X_i^2-2\bar{X}\boxed{\sum_{i=1}^{n}X_i}+n\bar{X}^2$$

$$=\sum_{i=1}^{n}X_i^2-2\bar{X}n\bar{X}+n\bar{X}^2$$

$$=\sum_{i=1}^{n}X_i^2-n\bar{X}^2$$

$$E(S^2)=E[\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2]$$

$$=E[\frac{1}{n-1}(\sum_{i=1}^{n}X_i^2-n\bar{X}^2)]$$

$$=\frac{1}{n-1}[\sum_{i=1}^{n}E(X_i^2)-nE(\bar{X}^2)]$$

$$=\frac{1}{n-1}[\sum_{i=1}^{n}(\sigma^2+\mu^2)-n(\frac{\sigma^2}{n}+\mu^2)]=\sigma^2$$

第39讲例2的结论并不是巧合，而是具有普遍性哦！

(2)因为总体$X\sim N\left(\mu,\sigma^2\right)$, 所以$\frac{(n-1)S^2}{\sigma^2}\sim\chi^2(n-1)$,

$$\Rightarrow D\left[\frac{(n-1)S^2}{\sigma^2}\right]=2(n-1)$$

$$\Rightarrow D(S^2)=\frac{2\sigma^4}{n-1}.$$

随样本量n增大，$D(S^2)$减小.