矩阵分块法

分块矩阵的概念

矩阵分块原则

分块矩阵的应用

分块矩阵的应用

设 A 是 n 阶方阵，

$$A=\begin{pmatrix} A_1 & & & O \\ & A_2 & & \\ & & \ddots & \\ O & & & A_s \end{pmatrix},$$

其中 $A_i\,(i=1,2,\cdots,s)$ 都是方阵，称 A 为分块对角阵.

性质： $|A|=|A_1|\cdot|A_2|\cdots|A_s|$. $|A|\neq 0 \Leftrightarrow |A_i|\neq 0$ $(i=1,2,\cdots,s)$.

并且

$$A^{-1} = \begin{pmatrix} A_1^{-1} & & & O \\ & A_2^{-1} & & \\ & & \ddots & \\ O & & & A_s^{-1} \end{pmatrix}.$$

例　设 $A=\left(\begin{array}{c|cc} 5 & 0 & 0 \\ \hline 0 & 3 & 1 \\ 0 & 2 & 1 \end{array}\right)$，求 A^{-1}.

解　$\boldsymbol{A}_1=(5)$，$\boldsymbol{A}_2=\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$，　$\boldsymbol{A}=\begin{pmatrix} \boldsymbol{A}_1 & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{A}_2 \end{pmatrix}$，

$$A^{-1}=\left(\begin{array}{c|cc} \frac{1}{5} & 0 & 0 \\ \hline 0 & 1 & -1 \\ 0 & -2 & 3 \end{array}\right), \boldsymbol{A}_1^{-1}=\left(\frac{1}{5}\right), \boldsymbol{A}_2^{-1}=\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}.$$

例 证明矩阵 $A=O$ 的充分必要条件是方阵 $A^{\mathrm{T}}A=O$.

证明 条件的必要性是显然的，下面证明充分性.

设 $A=\left(a_{ij}\right)_{m\times n}=\left(\alpha_1,\alpha_2,\cdots,\alpha_n\right)$, 则

$$A^{\mathrm{T}}A=\begin{pmatrix}\alpha_1^{\mathrm{T}}\\ \alpha_2^{\mathrm{T}}\\ \vdots\\ \alpha_n^{\mathrm{T}}\end{pmatrix}\left(\alpha_1,\alpha_2,\cdots,\alpha_n\right)=\begin{pmatrix}\underbrace{\alpha_1^{\mathrm{T}}\alpha_1}_{=0} & \alpha_1^{\mathrm{T}}\alpha_2 & \cdots & \alpha_1^{\mathrm{T}}\alpha_n\\ \alpha_2^{\mathrm{T}}\alpha_1 & \underbrace{\alpha_2^{\mathrm{T}}\alpha_2}_{=0} & \cdots & \alpha_2^{\mathrm{T}}\alpha_n\\ \vdots & \vdots & & \vdots\\ \alpha_n^{\mathrm{T}}\alpha_1 & \alpha_n^{\mathrm{T}}\alpha_2 & \cdots & \underbrace{\alpha_n^{\mathrm{T}}\alpha_n}_{=0}\end{pmatrix},$$

而

$$\alpha_j^{\mathrm{T}}\alpha_j=(a_{1j},a_{2j},\cdots,a_{mj})\begin{pmatrix}a_{1j}\\a_{2j}\\\vdots\\a_{mj}\end{pmatrix}=a_{1j}^2+a_{2j}^2+\cdots+a_{mj}^2=0,$$

得 $a_{1j}=a_{2j}=\cdots=a_{mj}=0\ (j=1,2,\cdots,n),$

即 $\boldsymbol{A}=\boldsymbol{O}.$

按列分块

按行分块

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{12}x_1 + a_{12}x_2 + \cdots + a_{2n}x_n = b_2, \\ \cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m, \end{cases} \Leftrightarrow A_{m\times n}x_{n\times 1} = b_{m\times 1},$$

$$A = (\alpha_1, \alpha_2, \cdots, \alpha_n),$$

$$\Leftrightarrow (\alpha_1, \alpha_2, \cdots, \alpha_n)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = b \Leftrightarrow x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n = b.$$