范德蒙行列式

定义：给定 n 个元素 $x_1, x_2, \cdots, x_n$，以其 $i-1$ 次幂 $x_1^i, x_2^i, \cdots, x_n^i$ 作为第 $i-1$ $(i=1,2,\cdots,n)$ 行得到的行列式，称为由 $x_1, x_2, \cdots, x_n$ 确定的 n 阶范德蒙行列式，记为 $V(x_1, x_2, \cdots, x_n)$

即：$$V(x_1, x_2, \cdots, x_n) = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}_n$$

结论：$$V(x_1, x_2, \cdots, x_n) = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}_n = \prod_{1 \le j < i \le n} (x_i - x_j)$$

这个结论说，由 $x_1, x_2, \cdots, x_n$ 确定的范德蒙行列式的值，等于所有不同的 $x_i - x_j \ (n \geq i > j \geq 1)$ 之积；也就是下列各项之积（注意观察规律）

$$
\begin{array}{llll}
(x_n - x_1) & (x_{n-1} - x_1) & \cdots & (x_3 - x_1) \quad (x_2 - x_1) \\
(x_n - x_2) & (x_{n-1} - x_2) & \cdots & (x_3 - x_2) \\
\cdots & \cdots & \cdots & \\
(x_n - x_{n-2}) & (x_{n-1} - x_{n-2}) & & \\
(x_n - x_{n-1}) & & &
\end{array}
$$

结论的证明：利用数学归纳法

当 $n=2$ 时，$V(x_1, x_2) = \begin{vmatrix} 1 & 1 \\ x_1 & x_2 \end{vmatrix} = x_2 - x_1$ ，结论成立；

假设当 $n=k$ 时，结论成立，即 $V(x_1,x_2,\cdots,x_k)=\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_k \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_k^2 \\ \vdots & \vdots & \vdots & & \vdots \\ x_1^{k-1} & x_2^{k-1} & x_3^{k-1} & \cdots & x_k^{k-1} \end{vmatrix}_k = \prod_{1\le j<i\le k}(x_i-x_j)$

当 $n=k+1$ 时，$V(x_1,x_2,\cdots,x_k,x_{k+1})=\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ x_1 & x_2 & x_3 & \cdots & x_k & x_{k+1} \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_k^2 & x_{k+1}^2 \\ \vdots & \vdots & \vdots & & \vdots & \\ x_1^{k-1} & x_2^{k-1} & x_3^{k-1} & \cdots & x_k^{k-1} & x_{k+1}^{k-1} \\ x_1^k & x_2^k & x_3^k & & x_k^k & x_{k+1}^k \end{vmatrix}_{k+1}$

将第 k 行乘 $-x_{k+1}$ 加到第 $k+1$ 行，得：$V(x_1,x_2,\cdots,x_k,x_{k+1})=\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ x_1 & x_2 & x_3 & \cdots & x_k & x_{k+1} \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_k^2 & x_{k+1}^2 \\ \vdots & \vdots & \vdots & & \vdots & \\ x_1^{k-1} & x_2^{k-1} & x_3^{k-1} & \cdots & x_k^{k-1} & x_{k+1}^{k-1} \\ x_1^{k-1}(x_1-x_{k+1}) & x_2^{k-1}(x_2-x_{k+1}) & x_3^{k-1}(x_3-x_{k+1}) & & x_k^{k-1}(x_k-x_{k+1}) & 0 \end{vmatrix}_{k+1}$

将第 i 行乘 $-x_{i+1}$ 加到第 $i+1$ 行，依次取 $i=k-1,k-2,\cdots,2,1$ 得：

$$V(x_1,x_2,\cdots,x_k,x_{k+1})=\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ x_1-x_{k+1} & x_2-x_{k+1} & x_3-x_{k+1} & \cdots & x_k-x_{k+1} & 0 \\ x_1(x_1-x_{k+1}) & x_2(x_2-x_{k+1}) & x_3(x_3-x_{k+1}) & \cdots & x_k(x_k-x_{k+1}) & 0 \\ \vdots & \vdots & \vdots & & \vdots & \\ x_1^{k-2}(x_1-x_{k+1}) & x_2^{k-2}(x_2-x_{k+1}) & x_3^{k-2}(x_3-x_{k+1}) & \cdots & x_k^{k-2}(x_k-x_{k+1}) & 0 \\ x_1^{k-1}(x_1-x_{k+1}) & x_2^{k-1}(x_2-x_{k+1}) & x_3^{k-1}(x_3-x_{k+1}) & & x_k^{k-1}(x_k-x_{k+1}) & 0 \end{vmatrix}_{k+1}$$

按第 $k+1$ 列展开，得

$$V(x_1,x_2,\cdots,x_k,x_{k+1})=(-1)^{k+1+1}\begin{vmatrix} x_1-x_{k+1} & x_2-x_{k+1} & x_3-x_{k+1} & \cdots & x_k-x_{k+1} \\ x_1(x_1-x_{k+1}) & x_2(x_2-x_{k+1}) & x_3(x_3-x_{k+1}) & \cdots & x_k(x_k-x_{k+1}) \\ \vdots & \vdots & \vdots & & \vdots \\ x_1^{k-2}(x_1-x_{k+1}) & x_2^{k-2}(x_2-x_{k+1}) & x_3^{k-2}(x_3-x_{k+1}) & \cdots & x_k^{k-2}(x_k-x_{k+1}) \\ x_1^{k-1}(x_1-x_{k+1}) & x_2^{k-1}(x_2-x_{k+1}) & x_3^{k-1}(x_3-x_{k+1}) & \cdots & x_k^{k-1}(x_k-x_{k+1}) \end{vmatrix}_k$$

第 $i\,(i=1,2,\cdots,k)$ 列分别提取公因式 x_i-x_{k+1} 得：

$$V(x_1,x_2,\cdots,x_k,x_{k+1})=(-1)^k(x_1-x_{k+1})(x_2-x_{k+1})\cdots(x_k-x_{k+1})\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_k \\ \vdots & \vdots & \vdots & & \vdots \\ x_1^{k-2} & x_2^{k-2} & x_3^{k-2} & \cdots & x_k^{k-2} \\ x_1^{k-1} & x_2^{k-1} & x_3^{k-1} & \cdots & x_k^{k-1} \end{vmatrix}_k=(x_{k+1}-x_k)\cdots(x_{k+1}-x_2)(x_{k+1}-x_1)V(x_1,x_2,\cdots,x_k)$$

利用归纳假设，得 $V(x_1,x_2,\cdots,x_k,x_{k+1})=\prod_{l=1}^{k}(x_{k+1}-x_l)\prod_{1\le j<i\le k}(x_i-x_j)=\prod_{1\le j<i\le k+1}(x_i-x_j)$ 　　证毕